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A METHOD OF TRANSVECTION IN THE ACTUAL COEFFICIENTS, AND AN APPLICATION TO EVECTANTS.

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The most common methods of obtaining, in actual coefficients, the concomitants of a binary algebraic form become almost impracticable in the case of forms of high order. If the concomitants are given in symbolic form, the process consists in performing tedious algebraic reductions, and combination of the symbols involved, in order to reduce them back to the coefficients which they were originally taken to represent. The other methods are processes of differentiation, the most important of which is the annihilator. The nature of the former method limits it to forms of low order, while the latter, which figures prominently in the methods of Cayley and Sylvester, involves the theories of partition of numbers and indeterminate analysis.

In this paper is briefly sketched a theory wherein every concomitant is represented as a summation. The advantages of the method are marked in the cases of the Aronhold and other invariant processes and in the theory of evectants. An interesting deduction from the theory is a proof that the complete system for a form is coextensive with transvectants of the form over other of its transvectants. This proof is not, however, included in the present note.

Consider two binary forms, of orders m and n , respectively, $m \geq n$:

$$f = a_x^m = a_0 x_1^m + m a_1 x_1^{m-1} x_2 + \dots$$

$$\phi = b_x^n = b_0 x_1^n + n b_1 x_1^{n-1} x_2 + \dots$$

If the symbol

$${}_p \left\{ \begin{matrix} q \\ \mu \end{matrix} \right\} {}_{st} \left\{ \begin{matrix} u \\ \nu \end{matrix} \right\} {}_w \equiv {}_p I_{st}^q I_w^u$$

be used to denote the following sum of binomial products

$$\left\{ \begin{matrix} q \\ p \end{matrix} \right\} \left\{ \begin{matrix} u \\ t \end{matrix} \right\} + \left\{ \begin{matrix} q \\ p+1 \end{matrix} \right\} \left\{ \begin{matrix} u \\ t+1 \end{matrix} \right\} + \dots + \left\{ \begin{matrix} q \\ s \end{matrix} \right\} \left\{ \begin{matrix} u \\ w \end{matrix} \right\},$$

then the r th transvectant of f over ϕ will be

$$(f, \phi)_{r=*} \binom{r}{\lambda} [{}_0 I_{k_0 0}^k I_{l_0 0}^l + {}_1 I_{k_0 0}^k I_{l_1 0}^l + {}_2 I_{k_0 0}^k I_{l_2 0}^l + \dots + {}_{n-r} I_{k_0 0}^k I_{l_{n-r} 0}^l + {}_{n-r+1} I_{k_1 0}^k I_{l_{n-r} 0}^l + \dots + {}_{m-r} I_{k_{m-n} 0}^k I_{l_{n-r} 0}^l + {}_{m-r} I_{k_{m-n+1} 1}^k I_{l_{n-r} 0}^l + \dots + {}_{m-r} I_{k_{m-r} n-r}^k I_{l_{n-r} 0}^l] a_{\mu+\lambda} b_{r+\nu-\lambda} (x_1 x_2)^{m+n-2r},$$

in which $k=m-r$, $l=n-r$.

*The signs of the numbers $\binom{r}{\lambda}$ alternate plus and minus.

Then if we let $[j-c]$ stand for the positive values (zero included) of the difference $j-c$ of two positive integers, as j takes successive values, the latter expression may be written*

$$(1) \quad (f, \phi)_r = c_x^{m+n-2r} \\ = \binom{r}{\lambda} \sum_{j=0}^{j=m+n-2r} \sum_{j-[j+r-m]} I_{[j+r-m]} I_{j-[j+r-m]} a_{\mu+\lambda} b_{r+\nu-\lambda} x_1^{m+n-2r-j} x_2^j.$$

It is to be noted that the subscript j is identical with the subscript of c in the term of (1) which a given j produces. This fact leads to an important modification in the cases when ϕ is a transvectant of f , or where both f and ϕ in (1) are replaced by transvectants.

Evidently we have

$$(f, f)_r = d_x^{2(m-r)} = \binom{r}{\lambda} \sum_{j=0}^{j=2(m-r)} \sum_{j-[j+r-m]} I_{[j+r-m]} I_{j-[j+r-m]} a_{\mu+\lambda} a_{r+\nu-\lambda} x_1^{2(m-r)-j} x_2^j.$$

$$\text{Then } (f, d)_s = e_x^{3m-2(r+s)} = (f, (f, f)_r)_s$$

$$= \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} l_1' I_{l_2', j'-l_1'} I_{j'-l_2'} a_{\mu'+\lambda'} \binom{r}{\lambda} \sum_{j=s+\nu'-\lambda'} l_1 I_{l_2 j-l_1} I_{j-l_2} a_{\mu+\lambda} a_{r+\nu-\lambda} x_1^{3m-2(r+s)-j'} x_2^{j'}.$$

$$\text{where } l_1' = j' - [j+s-m], l_1 = j - [j+r-m], l_2' = [j'+s+2r-2m], l_2 = [j+r-m].$$

Replacing j by its value, we get

$$(f, d)_s = \binom{s}{\lambda'} \sum_{j'=0}^{j'=3m-2(r+s)} l_1' I_{l_2', j'-l_1'} I_{j'-l_2'} a_{\mu'+\lambda'} \binom{r}{\lambda} k_1 - l_1' I_{l_1''} l_1' I_{k_1 - l_1''} \\ \times a_{\mu+\lambda} a_{r+\nu-\lambda} x_1^{3m-2(r+s)-j'} x_2^{j'}.$$

Finally, making use of the commutative properties, we write this

$$\binom{s}{\lambda'} \binom{r}{\lambda} \sum_{j'=0}^{j'=3m-2(r+s)} l_1' I_{l_2', j'-l_1'} I_{j'-l_2'}; k_1 - l_1' I_{l_1''}, l_1' I_{k_1 - l_1''} \\ \times a_{\mu'+\lambda'} a_{\mu+\lambda} a_{r+\nu-\lambda} x_1^{3m-2(r+s)-j'} x_2^{j'}$$

where $k_1 = s + \nu' - \lambda'$, $l_1' = l_1'' = [k + r - m]$.

The t th transvectant of f over this covariant will be

*The superscript to I is omitted, being fully determinate from the subscripts.

$$(f, l)_t = \binom{t}{\lambda'} \binom{s}{\lambda} \sum_{j''=0}^{j''=4m-2(t+r+s)} i_1' I_{i_2'', j''-i_1'} I_{j''-i_2''} I_{k_2-l_2'} I_{l_2''} I_{k_2-l_2''} I_{k_1-l_1'} I_{l_1''} I_{k_1-l_1''} \\ \times a_{\mu'+\lambda'} a_{\mu'+\lambda'} a_{\mu+\lambda} a_{r+\nu-\lambda} x_1^{4m-2(r+s+t)-j''} x_2^{j''}.$$

in which

$$i''_1 = j'' - [j'' + r - m], \quad k_2 = t + \nu'' - \lambda'', \\ i''_2 = [j'' + t + 2r + 2s - 3m], \quad l'_2 = [k_2 + s - m], \quad l''_2 = [k_2 + s + 2r - 2m].$$

In general, we will have therefore,

$$(2) \quad (f, \dots (f, (f, (f, f)_r)_s)_t \dots)_z = \\ j^{(h)} = (h+2)m - 2(r+s+\dots+z) \\ \left\{ \begin{matrix} z \\ \lambda^{(h)} \end{matrix} \right\} \left\{ \begin{matrix} y \\ \lambda^{(h-1)} \end{matrix} \right\} \dots \left\{ \begin{matrix} s \\ \lambda' \end{matrix} \right\} \left\{ \begin{matrix} r \\ \lambda \end{matrix} \right\} \sum_{j^{(h)}=0}^{j^{(h)}=(h+2)m-2(r+s+\dots+z)} i_{1,h} I_{i_2^{(h)}, j^{(h)}-i_{1,h}} I_{j^{(h)}-i_2^{(h)}} I_{k_h-l'_h} I_{l'_h} I_{k_h-l''_h} \dots \\ \dots k_{1-l'_1} I_{l'_1} I_{k_1-l'_1} a_{\mu^{(h)}+\lambda^{(h)}} a_{\mu^{(h-1)}+\lambda^{(h-1)}} \dots a_{\mu'+\lambda'} a_{\mu+\lambda} a_{r+\nu-\lambda} \\ \times x_1^{(h+2)m-2(r+\dots+z)-j^{(h)}} x_2^{j^{(h)}}.$$

This is a covariant of order $(h+2)m - 2(r+s+\dots+z)$ degree $(h+2)$, weight $(h+2)m - (r+s+\dots+z)$ and index $(r+s+t+\dots+z)$.

Any concomitant in the series obtained by giving to h all integral values from zero to $\frac{2}{m}(r+s+\dots+z-m)$, inclusive, may be obtained from the summation, directly, without knowledge of the concomitants which precede.

If $h = \frac{2}{m}(r+s+\dots+z-m)$, (2) is an *invariant* and there being but one term, $j^{(h)}=0$, the summation sign may be removed, giving,

$$(3) \quad \left\{ \begin{matrix} z \\ \lambda^{(h)} \end{matrix} \right\} \left\{ \begin{matrix} y \\ \lambda^{(h-1)} \end{matrix} \right\} \dots \left\{ \begin{matrix} s \\ \lambda' \end{matrix} \right\} \left\{ \begin{matrix} r \\ \lambda \end{matrix} \right\} k_h-l'_h I_{l'_h} I_{k_h-l'_h} \dots k_1-l'_1 I_{l'_1} I_{k_1-l'_1} \\ \times a_{\lambda^{(h)}} a_{\mu^{(h-1)}+\lambda^{(h-1)}} \dots a_{\mu'+\lambda'} a_{r+\nu-\lambda}.$$

For the sake of illustration, a few concomitants in actual coefficients are obtained.

Let $f = a_x^3 = a_0 x_1^3 + 3a_1 x_1^2 x_2 + \dots$, $\phi = b_x^2 = b_0 x_1^2 + 2b_1 x_1 x_2 + \dots$. Then

$$(f, \phi) = \binom{1}{\lambda} \sum_{j=0}^{j=3} i_1 I_{i_2, j-i_1} I_{j-i_2} a_{\mu+\lambda} b_{1+\nu-\lambda} x_1^{3-j} x_2^j \quad (i_2 = j - \binom{j-2}{i_1}) = c_x^3 \\ = (a_0 b_1 - a_1 b_0) x_1^3 + [2(a_1 b_1 - a_2 b_0) + (a_0 b_2 - a_1 b_1)] x_1^2 x_2 + [(a_2 b_1 - a_3 b_0) \\ + (a_1 b_2 - a_2 b_1)] x_1 x_2^2 + (a_2 b_2 - a_3 b_1) x_2^3.$$

$$(f, c)_3 = \binom{3}{\lambda} a_{\lambda} \binom{1}{\lambda} k_1-l'_1 I_{l'_1} I_{k_1-l'_1} a_{\mu+\lambda} b_{1+\nu-\lambda} \\ = a_0 (a_2 b_2 - a_3 b_1) - a_1 (a_1 b_2 - a_2 b_1) + a_2 (a_0 b_2 - 2a_2 b_0 + a_1 b_1) - a_3 (a_0 b_1 - a_1 b_0).$$

Let next $f = a_x^4 = a_0 x_1^4 + 4a_1 x_1^3 x_2 + \dots$

$$\therefore (f, f)_4 = \binom{4}{\lambda} a_{\lambda} a_{4-\lambda} = 2(a_0 a_4 - 4a_1 a_3 + 3a_2^2) = i, \\ (f, (f, f)_2)_4 = \binom{4}{\lambda} a_{\lambda} \binom{2}{\lambda} k_1-l'_1 I_{l'_1} I_{k_1-l'_1} a_{\mu+\lambda} a_{2+\nu-\lambda}$$

$$\begin{aligned}
&= 2a_0(a_2a_4 - a_3^2) - a_1[2(a_2a_3 - 2a_3a_2 + a_4a_1) + 2(a_1a_4 - 2a_2a_3 + a_3a_2)] \\
&+ a_2[(a_2^2 - 2a_3a_1 + a_4a_0) + 4(a_1a_3 - 2a_2^2 + a_3a_1) + (a_0a_4 - 2a_1a_3 + a_2^2)] \\
&- a_3[2(a_1a_2 - 2a_2a_1 + a_3a_0) + 2(a_0a_3 - 2a_1a_2 + a_2a_1)] + 2a_4(a_0a_2 - a_1^2) \\
&= 6(a_0a_2a_4 + 2a_1a_2a_3 - a_2^3 - a_0a_3^2 - a_1^2a_4) = J.
\end{aligned}$$

The non-vanishing (even) transvectants of the binary 11^{-io} over itself are five in number. Of the forty-two transvectants of the 11^{-io} , $f=0$, over these five, viz.,

$$F, G, \dots, Z, A', B', \dots, R', S',$$

K' is linear (evanescent) and N, W, F', N' and R' are of order 11. The 11th transvectants of f over these latter five give five invariants I_1, \dots, I_5 . These may be given in their simplified form. We have*

$$\begin{aligned}
E &= (f, f)_{10} = \sum_{j=0}^{j=2} \sum_{\lambda} \binom{10}{\lambda} I_{[j-1]} I_{[j-1]} I_{j-[j-1]} a_{\mu+\lambda} a_{10+\nu-\lambda} x_1^{2-j} x_2^j \\
&= (a_0a_{10} - 20a_1a_9 + 90a_2a_8 - 240a_3a_7 + 420a_4a_6 - 252a_5^2)x_1^2 \\
&+ 2(a_0a_{11} - 9a_1a_{10} + 35a_2a_9 - 75a_3a_8 + 90a_4a_7 - 42a_5a_6)x_1x_2 \\
&+ (2a_1a_{11} - 20a_2a_{10} + 90a_3a_9 - 240a_4a_8 + 420a_5a_7 - 252a_6^2)x_2^2.
\end{aligned}$$

$$\begin{aligned}
I_5 &= (f, (f, (f, f)_{10})_1)_{11} = \binom{11}{\lambda} a_{\lambda'} \binom{1}{\lambda} a_{k_2-l_2} I_{l_2-l_2} I_{l_2-l_2} a_{\mu'+\lambda'} \binom{10}{\lambda} a_{k_1-l_1} I_{l_1-l_1} I_{k_1-l_1} \\
&\quad \times a_{\mu+\lambda} a_{10+\nu-\lambda}
\end{aligned}$$

$$\begin{aligned}
&= -2a_0^2a_{11}^2 + 44a_0a_1a_{10}a_{11} - 80a_0a_2a_{10}^2 + 360a_0a_3a_9a_{10} - 960a_0a_4a_8a_{10} \\
&+ 1680a_0a_5a_7a_{10} - 1008a_0a_6^2a_{10} - 140a_0a_2a_9a_{11} + 300a_0a_3a_8a_{11} \\
&- 360a_0a_4a_7a_{11} + 168a_0a_5a_6a_{11} - 162a_1^2a_{10}^2 + 2060a_1a_2a_9a_{10} - 2700a_1a_3a_8a_{10} \\
&+ 3240a_1a_4a_7a_{10} - 1512a_1a_5a_6a_{10} - 80a_1^2a_9a_{11} + 360a_1a_2a_8a_{11} - 960a_1a_3a_7a_{11} \\
&+ 1680a_1a_4a_6a_{11} - 1008a_1a_5^2a_{11} - 3600a_1a_3a_9^2 + 9600a_1a_4a_8a_9 - 1680a_1a_5a_7a_9 \\
&+ 10080a_1a_6^2a_9 - 2450a_2^2a_9^2 + 26700a_2a_3a_8a_9 - 12600a_2a_4a_7a_9 + 5880a_2a_5a_6a_9 \\
&- 3600a_2^2a_8a_{10} + 9600a_2a_3a_7a_{10} - 16800a_2a_4a_6a_{10} + 10080a_2a_5^2a_{10} \\
&- 43200a_2a_4a_8^2 + 75600a_2a_5a_7a_8 - 45360a_2a_6^2a_8 - 11250a_3^2a_8^2 \\
&+ 142200a_3a_4a_7a_8 - 12600a_3a_5a_6a_8 - 43200a_3^2a_7^2 + 75600a_3a_4a_6a_9 \\
&- 45360a_3a_5^2a_9 - 201600a_3a_5a_7^2 + 120960a_3a_6^2a_7 - 16200a_4^2a_7^2 \\
&+ 367920a_4a_5a_6a_7 - 201600a_4^2a_6a_8 + 120960a_4a_5^2a_8 - 211680a_4a_6^3 \\
&+ 123480a_5^2a_6^2 - 211680a_5^3a_7.
\end{aligned}$$

Index=weight=22.

$$\text{Now } (E_x^2, E_x^2)_2 = ((f, f)_{10}, (f, f)_{10})_2 = D_{E_x^2}$$

$$\begin{aligned}
&= \binom{2}{\lambda'} \binom{10}{\lambda} a_{\lambda'-[\lambda'-1]} I_{[\lambda'-1]} I_{[\lambda'-1]} I_{\lambda'-[\lambda'-1]} a_{\mu+\lambda} a_{10+\nu-\lambda} \binom{10}{\lambda} a_{2-\lambda'-[1-\lambda']} I_{[1-\lambda']} I_{[1-\lambda']} I_{2-\lambda'-[1-\lambda']} \\
&\quad \times a_{\mu+\lambda} a_{10+\nu-\lambda}.
\end{aligned}$$

But $\binom{2}{\lambda'} \binom{10}{\lambda} = \binom{1}{\lambda'} \binom{11}{\lambda}$. Hence there immediately results $I_5 = D_{E_x^2}$.

[To be completed in the May number of the MONTHLY.]

*In a memoir on the octic form, in preparation, I give for the first time the complete system for the octic in actual coefficients.